

Self-Dual Abelian Codes in some Non-Principal Ideal Group Algebras

Parinyawat Choosuwan, Somphong Jitman, and Patanee Udomkavanich

Abstract

The main focus of this paper is the complete enumeration of self-dual abelian codes in non-principal ideal group algebras $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ with respect to both the Euclidean and Hermitian inner products, where k and s are positive integers and A is an abelian group of odd order. Based on the well-known characterization of Euclidean and Hermitian self-dual abelian codes, we show that such enumeration can be obtained in terms of a suitable product of the number of cyclic codes, the number of Euclidean self-dual cyclic codes, and the number of Hermitian self-dual cyclic codes of length 2^s over some Galois extensions of the ring $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$, where $u^2 = 0$. Subsequently, general results on the characterization and enumeration of cyclic codes and self-dual codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are given. Combining these results, the complete enumeration of self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ is therefore obtained.

Keywords: self-dual codes, abelian codes, finite chain rings, group algebras

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1 Introduction

Information media, such as communication systems and storage devices of data, are not 100 percent reliable in practice because of noise or other forms of introduced interference. The art of error correcting codes is a branch of Mathematics that has been introduced to deal with this problem since 1960s. Linear codes with additional algebraic structures

P. Choosuwan is with the Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand. Email: parinyawat.ch@gmail.com

S. Jitman is with the Department of Mathematics, Faculty of Science, Silpakorn University, Nakhon Pathom 73000, Thailand. Email: sjitman@gmail.com

P. Udomkavanich is with the Department of Mathematics and Computer Science, Faculty of Science, Chulalongkorn University, Bangkok 10330, Thailand. Email: pattanee.u@chula.ac.th

and self-dual codes are important families of codes that have been extensively studied for both theoretical and practical reasons (see [2–6, 14, 16, 21–23] and references therein). Some major results on Euclidean self-dual cyclic codes have been established in [18]. In [13], the complete characterization and enumeration of such self-dual codes have been given. These results on Euclidean self-dual cyclic codes have been generalized to abelian codes in group algebras [14] and the complete characterization and enumeration of Euclidean self-dual abelian codes in principal ideal group algebras (PIGAs) have been established. Extensively, the characterization and enumeration of Hermitian self-dual abelian codes in PIGAs have been studied in [16]. To the best of our knowledge, the characterization and enumeration of self-dual abelian codes in non-principal ideal group algebras (non-PIGAs) have not been well studied. It is therefore of natural interest to focus on this open problem.

In [14] and [16], it has been shown that there exists a Euclidean (resp., Hermitian) self-dual abelian code in $\mathbb{F}_{p^k}[G]$ if and only if $p = 2$ and $|G|$ is even. In order to study self-dual abelian codes, it is therefore restricted to the group algebra $\mathbb{F}_{2^k}[A \times B]$, where A is an abelian group of odd order and B is a non-trivial abelian group of two power order. In this case, $\mathbb{F}_{2^k}[A \times B]$ is a PIGA if and only if $B = \mathbb{Z}_{2^s}$ is a cyclic group (see [12]). Equivalently, $\mathbb{F}_{2^k}[A \times B]$ is a non-PIGA if and only if B is non-cyclic. To avoid tedious computations, we focus on the simplest case where $B = \mathbb{Z}_2 \times \mathbb{Z}_{2^s}$, where s is a positive integer. Precisely, the goal of this paper is to determine the algebraic structures and the numbers of Euclidean and Hermitian self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$.

It turns out that every Euclidean (resp., Hermitian) self-dual abelian code in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ is a suitable Cartesian product of cyclic codes, Euclidean self-dual cyclic codes, and Hermitian self-dual cyclic codes of length 2^s over some Galois extension of the ring $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$, where $u^2 = 0$ (see Section 2). Hence, the number of self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ can be determined in terms of the cyclic codes mentioned earlier. Subsequently, useful properties of cyclic codes, Euclidean self-dual cyclic codes, and Hermitian self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are given for all primes p . Combining these results, the characterizations and enumerations of Euclidean and Hermitian self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ are rewarded.

The paper is organized as follows. In Section 2, some basic results on abelian codes are recalled together with a link between abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ and cyclic codes of length 2^s over Galois extensions of $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$. General results on the characterization and enumeration of cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are provided in Section 3. In Section 4, the characterizations and enumerations of Euclidean and Hermitian self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are established. Summary and remarks are given in Section 5.

2 Preliminaries

In this section, we recall some definitions and basic properties of rings and abelian codes. Subsequently, a link between an abelian code in non-principal ideal algebras and a product of cyclic codes over rings is given. This link plays an important role in determining the algebraic structures and the numbers of Euclidean and Hermitian self-dual abelian codes in non-principal ideal algebras.

2.1 Rings and Abelian Codes in Group Rings

For a prime p and a positive integer k , denote by \mathbb{F}_{p^k} the finite field of order p^k . Let $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k} := \{a + ub \mid a, b \in \mathbb{F}_{p^k}\}$ be a ring, where the addition and multiplication are defined as in the usual polynomial ring over \mathbb{F}_{p^k} with indeterminate u together with the condition $u^2 = 0$. We note that $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ is isomorphic to $\mathbb{F}_{p^k}[u]/\langle u^2 \rangle$ as rings. The Galois extension of $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ of degree m is defined to be the quotient ring $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle f(x) \rangle$, where $f(x)$ is an irreducible polynomial of degree m over \mathbb{F}_{p^k} . It is not difficult to see that the Galois extension of $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ of degree m is isomorphic to $\mathbb{F}_{p^{km}} + u\mathbb{F}_{p^{km}}$ as rings. In the case where k is even, the mapping $a + ub \mapsto a^{p^{k/2}} + ub^{p^{k/2}}$ is a ring automorphism of order 2 on $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$. The readers may refer to [7, 8] for properties of the ring $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$.

For a commutative ring R with identity 1 and a finite abelian group G , written additively, let $R[G]$ denote the *group ring* of G over R . The elements in $R[G]$ will be written as $\sum_{g \in G} \alpha_g Y^g$, where $\alpha_g \in R$. The addition and the multiplication in $R[G]$ are given as in the usual polynomial ring over R with indeterminate Y , where the indices are computed additively in G . Note that $Y^0 := 1$ is the multiplicative identity of $R[G]$ (resp., R), where 0 is the identity of G . We define a *conjugation* $\bar{\cdot}$ on $R[G]$ to be the map that fixes R and sends Y^g to Y^{-g} for all $g \in G$, i.e., for $\mathbf{u} = \sum_{g \in G} \alpha_g Y^g \in R[G]$, we set $\bar{\mathbf{u}} := \sum_{g \in G} \alpha_g Y^{-g} = \sum_{g \in G} \alpha_{-g} Y^g$. In the case where, there exists a ring automorphism ρ on R of order 2, we define $\tilde{\mathbf{u}} := \sum_{g \in G} \rho(\alpha_g) Y^{-g}$ for all $\mathbf{u} = \sum_{g \in G} \alpha_g Y^g \in R[G]$. In the case where R is a finite field \mathbb{F}_{p^k} , then $\mathbb{F}_{p^k}[G]$ can be viewed as an \mathbb{F}_{p^k} -algebra and it is called a *group algebra*. The group algebra $\mathbb{F}_{p^k}[G]$ is called a *principal ideal group algebra (PIGA)* if its ideals are generated by a single element.

An *abelian code* in $R[G]$ is defined to be an ideal in $R[G]$. If G is cyclic, this code is called a *cyclic code*, a code which is invariant under the right cyclic shift. It is well known that cyclic codes of length n over R can be regarded as ideals in the quotient polynomial ring $R[x]/\langle x^n - 1 \rangle \cong R[\mathbb{Z}_n]$.

The *Euclidean inner product* in $R[G]$ is defined as follows. For

$$\mathbf{u} = \sum_{g \in G} \alpha_g Y^g \text{ and } \mathbf{v} = \sum_{g \in G} \beta_g Y^g$$

in $R[G]$, we set

$$\langle \mathbf{u}, \mathbf{v} \rangle_E := \sum_{g \in G} \alpha_g \beta_g.$$

In addition, if there exists a ring automorphism ρ of order 2 on R , the ρ -*inner product* of \mathbf{u} and \mathbf{v} is defined by

$$\langle \mathbf{u}, \mathbf{v} \rangle_\rho := \sum_{g \in G} \alpha_g \rho(\beta_g).$$

If $R = \mathbb{F}_{q^2}$ (resp., $R = \mathbb{F}_{q^2} + u\mathbb{F}_{q^2}$) and $\rho(a) = a^q$ (resp., $\rho(a + ub) = a^q + ub^q$) for all $a \in \mathbb{F}_{q^2}$ (resp., $a + ub \in \mathbb{F}_{q^2} + u\mathbb{F}_{q^2}$), the ρ -inner product is called the *Hermitian inner product* and denoted by $\langle \mathbf{u}, \mathbf{v} \rangle_H$.

The *Euclidean dual* and *Hermitian dual* of C in $R[G]$ are defined to be the sets

$$C^{\perp_E} := \{\mathbf{u} \in R[G] \mid \langle \mathbf{u}, \mathbf{v} \rangle_E = 0 \text{ for all } \mathbf{v} \in C\}$$

and

$$C^{\perp_H} := \{\mathbf{u} \in R[G] \mid \langle \mathbf{u}, \mathbf{v} \rangle_H = 0 \text{ for all } \mathbf{v} \in C\},$$

respectively.

An abelian code C is said to be Euclidean self-dual (resp., Hermitian self-dual) if $C = C^{\perp_E}$ (resp., $C = C^{\perp_H}$).

For convenience, denote by $N(p^k, n)$, $NE(p^k, n)$, and $NH(p^k, n)$ the number of cyclic codes, the number of Euclidean self-dual cyclic codes, and the number of Hermitian self-dual cyclic codes of length n over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$, respectively.

2.2 Decomposition of Abelian Codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$

In [14] and [16], it has been shown that there exists a Euclidean (resp., Hermitian) self-dual abelian code in $\mathbb{F}_{p^k}[G]$ if and only if $p = 2$ and $|G|$ is even. To study self-dual abelian codes, it is sufficient to focus on $\mathbb{F}_{2^k}[A \times B]$, where A is an abelian group of odd order and B is a non-trivial abelian group of two power order. In this case, $\mathbb{F}_{2^k}[A \times B]$ is a PIGA if and only if $B = \mathbb{Z}_{2^s}$ is a cyclic group for some positive integer s (see [12]). The complete characterization and enumeration of self-dual abelian codes in PIGAs have been given in [14] and [16]. Here, we focus on self-dual abelian codes in non-PIGAs, or equivalently, B is non-cyclic. To avoid tedious computations, we establish results for

the simplest case where $B = \mathbb{Z}_2 \times \mathbb{Z}_{2^s}$. Useful decompositions of $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ are given in this section.

First, we consider the decomposition of $\mathcal{R} := \mathbb{F}_{2^k}[A]$. In this case, \mathcal{R} is semi-simple [2] which can be decomposed using the Discrete Fourier Transform in [22] (see [16] and [14] for more details). For completeness, the decompositions used in this paper are summarized as follows.

For an odd positive integer i and a positive integer k , let $\text{ord}_i(2^k)$ denote the multiplicative order of 2^k modulo i . For each $a \in A$, denote by $\text{ord}(a)$ the additive order of a in A . A 2^k -cyclotomic class of A containing $a \in A$, denoted by $S_{2^k}(a)$, is defined to be the set

$$S_{2^k}(a) := \{2^{ki} \cdot a \mid i = 0, 1, \dots\} = \{2^{ki} \cdot a \mid 0 \leq i < \text{ord}_{\text{ord}(a)}(2^k)\},$$

where $2^{ki} \cdot a := \sum_{j=1}^{2^{ki}} a$ in A .

An *idempotent* in \mathcal{R} is a non-zero element e such that $e^2 = e$. It is called *primitive* if for every other idempotent f , either $ef = e$ or $ef = 0$. The primitive idempotents in \mathcal{R} are induced by the 2^k -cyclotomic classes of A (see [6, Proposition II.4]). Let $\{a_1, a_2, \dots, a_t\}$ be a complete set of representatives of 2^k -cyclotomic classes of A and let e_i be the primitive idempotent induced by $S_{2^k}(a_i)$ for all $1 \leq i \leq t$. From [22], \mathcal{R} can be decomposed as

$$\mathcal{R} = \bigoplus_{i=1}^t \mathcal{R}e_i, \quad (2.1)$$

and hence,

$$\mathbb{F}_{2^k}[A \times \mathbb{Z}_2] \cong \mathcal{R}[\mathbb{Z}_2] \cong \bigoplus_{i=1}^t (\mathcal{R}e_i)[\mathbb{Z}_2]. \quad (2.2)$$

It is well known (see [14, 16]) that $\mathcal{R}e_i := \mathbb{F}_{2^{k_i}}$, where k_i is a multiple of k . Precisely, $k_i = k|S_{2^k}(a_i)| = k \cdot \text{ord}_{\text{ord}(a_i)}(2^k)$ provided that e_i is induced by $S_{2^k}(a_i)$. It follows that $\mathcal{R}e_i[\mathbb{Z}_2] \cong \mathbb{F}_{2^{k_i}}[\mathbb{Z}_2]$. Under the ring isomorphism that fixes the elements in $\mathbb{F}_{2^{k_i}}$ and $Y^1 \mapsto u + 1$, $\mathbb{F}_{2^{k_i}}[\mathbb{Z}_2]$ is isomorphic to the ring $\mathbb{F}_{2^{k_i}} + u\mathbb{F}_{2^{k_i}}$, where $u^2 = 0$. We note that this ring plays an important role in coding theory and codes over rings in this family have extensively been studied [7, 8, 10, 17] and references therein.

From (2.2) and the ring isomorphism discussed above, we have

$$\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}] \cong \prod_{i=1}^t \mathcal{R}_i[\mathbb{Z}_{2^s}], \quad (2.3)$$

where $\mathcal{R}_i := \mathbb{F}_{2^{k_i}} + u\mathbb{F}_{2^{k_i}}$ for all $1 \leq i \leq t$.

In order to study the algebraic structures of Euclidean and Hermitian self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$, the two rearrangements of \mathcal{R}_i 's in the decomposition (2.3) are needed. The details are given in the following two subsections.

2.2.1 Euclidean Case

A 2^k -cyclotomic class $S_{2^k}(a)$ is said to be of *type I* if $a = -a$ (in this case, $S_{2^k}(a) = S_{2^k}(-a)$), *type II* if $S_{2^k}(a) = S_{2^k}(-a)$ and $a \neq -a$, or *type III* if $S_{2^k}(-a) \neq S_{2^k}(a)$. The primitive idempotent e induced by $S_{2^k}(a)$ is said to be of type $\lambda \in \{I, II, III\}$ if $S_{2^k}(a)$ is a 2^k -cyclotomic class of type λ .

Without loss of generality, the representatives a_1, a_2, \dots, a_t of 2^k -cyclotomic classes of A can be chosen such that $\{a_i \mid i = 1, 2, \dots, r_I\}$, $\{a_{r_I+j} \mid j = 1, 2, \dots, r_{II}\}$ and $\{a_{r_I+r_{II}+l}, a_{r_I+r_{II}+r_{III}+l} = -a_{r_I+r_{II}+l} \mid l = 1, 2, \dots, r_{III}\}$ are sets of representatives of 2^k -cyclotomic classes of types *I*, *II*, and *III*, respectively, where $t = r_I + r_{II} + 2r_{III}$.

Rearranging the terms in the decomposition of \mathcal{R} in (2.1) based on the 3 types primitive idempotents, we have

$$\mathbb{F}_{2^k}[A \times \mathbb{Z}_2] \cong \bigoplus_{i=1}^t (\mathcal{R}e_i)[\mathbb{Z}_2] \cong \left(\prod_{i=1}^{r_I} \mathcal{R}_i \right) \times \left(\prod_{j=1}^{r_{II}} \mathcal{S}_j \right) \times \left(\prod_{l=1}^{r_{III}} (\mathcal{T}_l \times \mathcal{T}_l) \right), \quad (2.4)$$

where $\mathcal{R}_i := \mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$ for all $i = 1, 2, \dots, r_I$, $\mathcal{S}_j := \mathbb{F}_{2^{k_{r_I+j}}} + u\mathbb{F}_{2^{k_{r_I+j}}}$ for all $j = 1, 2, \dots, r_{II}$, and $\mathcal{T}_l := \mathbb{F}_{2^{k_{r_I+r_{II}+l}}} + u\mathbb{F}_{2^{k_{r_I+r_{II}+l}}}$ for all $l = 1, 2, \dots, r_{III}$.

From (2.4), we have

$$\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}] \cong \left(\prod_{i=1}^{r_I} \mathcal{R}_i[\mathbb{Z}_{2^s}] \right) \times \left(\prod_{j=1}^{r_{II}} \mathcal{S}_j[\mathbb{Z}_{2^s}] \right) \times \left(\prod_{l=1}^{r_{III}} (\mathcal{T}_l[\mathbb{Z}_{2^s}] \times \mathcal{T}_l[\mathbb{Z}_{2^s}]) \right). \quad (2.5)$$

It follows that, an abelian code C in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ can be viewed as

$$C \cong \left(\prod_{i=1}^{r_I} B_i \right) \times \left(\prod_{j=1}^{r_{II}} C_j \right) \times \left(\prod_{l=1}^{r_{III}} (D_l \times D'_l) \right), \quad (2.6)$$

where B_i , C_j , D_s and D'_s are cyclic codes in $\mathcal{R}_i[\mathbb{Z}_{2^s}]$, $\mathcal{S}_j[\mathbb{Z}_{2^s}]$, $\mathcal{T}_l[\mathbb{Z}_{2^s}]$ and $\mathcal{T}_l[\mathbb{Z}_{2^s}]$, respectively, for all $i = 1, 2, \dots, r_I$, $j = 1, 2, \dots, r_{II}$ and $l = 1, 2, \dots, r_{III}$.

Using the analysis similar to those in [14, Section II.D], the Euclidean dual of C in (2.6) is of the form

$$C^{\perp_E} \cong \left(\prod_{i=1}^{r_I} B_i^{\perp_E} \right) \times \left(\prod_{j=1}^{r_{II}} C_j^{\perp_H} \right) \times \left(\prod_{l=1}^{r_{III}} ((D'_l)^{\perp_E} \times D_l^{\perp_E}) \right).$$

Similar to [14, Corollary 2.9], necessary and sufficient conditions for an abelian code in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ to be Euclidean self-dual can be given using the notions of cyclic codes of length 2^s over \mathcal{R}_i , \mathcal{S}_j , and \mathcal{T}_l in the following corollary.

Corollary 2.1. *An abelian code C in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ is Euclidean self-dual if and only if in the decomposition (2.6),*

- i) B_i is a Euclidean self-dual cyclic code of length 2^s over \mathcal{R}_i for all $i = 1, 2, \dots, r_I$,
- ii) C_j is a Hermitian self-dual cyclic code of length 2^s over \mathcal{S}_j for all $j = 1, 2, \dots, r_{II}$,
and
- iii) $D'_l = D_l^{\perp_E}$ is a cyclic code of length 2^s over \mathcal{T}_l for all $l = 1, 2, \dots, r_{III}$.

Given a positive integer k and an odd positive integer j , the pair $(j, 2^k)$ is said to be *good* if j divides $2^{kt} + 1$ for some integer $t \geq 1$, and *bad* otherwise. These notions have been introduced in [13, 14] for the enumeration of self-dual cyclic codes and self-dual abelian codes over finite fields.

Let χ be a function defined on the pair $(j, 2^k)$, where j is an odd positive integer, as follows.

$$\chi(j, 2^k) = \begin{cases} 0 & \text{if } (j, 2^k) \text{ is good,} \\ 1 & \text{otherwise.} \end{cases} \quad (2.7)$$

The number of Euclidean self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ can be determined as follows.

Theorem 2.2. *Let k and s be positive integers and let A be a finite abelian group of odd order and exponent M . Then the number of Euclidean self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ is*

$$\begin{aligned} & \sum_{d|M, \text{ord}_d(2^k)=1}^{(1-\chi(d, 2^k))\mathcal{N}_A(d)} (NE(2^k, 2^s))^{d|M, \text{ord}_d(2^k)=1} \times \prod_{\substack{d|M \\ \text{ord}_d(2^k) \neq 1}} \left(NH(2^{k \cdot \text{ord}_d(2^k)}, 2^s) \right)^{(1-\chi(d, 2^k)) \frac{\mathcal{N}_A(d)}{\text{ord}_d(2^k)}} \\ & \times \prod_{d|M} \left(N(2^{k \cdot \text{ord}_d(2^k)}, 2^s) \right)^{\chi(d, 2^k) \frac{\mathcal{N}_A(d)}{2 \text{ord}_d(2^k)}}, \end{aligned}$$

where $\mathcal{N}_A(d)$ denotes the number of elements in A of order d determined in [1].

Proof. From (2.6) and Corollary 2.1, it suffices to determine the numbers of cyclic codes B_i 's, C_j 's, and D_l 's such that B_i and C_j are Euclidean and Hermitian self-dual, respectively.

From [16, Remark 2.5], the elements in A of the same order are partitioned into 2^k -cyclotomic classes of the same type. For each divisor d of M , a 2^k -cyclotomic class containing an element of order d has cardinality $\text{ord}_d(2^k)$ and the number of such 2^k -cyclotomic classes is $\frac{\mathcal{N}_A(d)}{\text{ord}_d(2^k)}$. We consider the following 3 cases.

Case 1: $\chi(d, 2^k) = 0$ and $\text{ord}_d(2^k) = 1$. By [14, Remark 2.6], every 2^k -cyclotomic class of A containing an element of order d is of type *I*. Since there are $\frac{\mathcal{N}_A(d)}{\text{ord}_d(2^k)}$ such 2^k -cyclotomic classes, the number of Euclidean self-dual cyclic codes B_i 's of length 2^s corresponding to d is

$$(NE(2^k, 2^s))^{\frac{\mathcal{N}_A(d)}{\text{ord}_d(2^k)}} = (NE(2^k, 2^s))^{(1-\chi(d, 2^k))\mathcal{N}_A(d)}.$$

Case 2: $\chi(d, 2^k) = 0$ and $\text{ord}_d(2^k) \neq 1$. By [14, Remark 2.6], every 2^k -cyclotomic class of A containing an element of order d is of type *II*. Hence, the number of Hermitian self-dual cyclic codes C_j 's of length 2^s corresponding to d is

$$(NH(2^{k \cdot \text{ord}_d(2^k)}, 2^s))^{\frac{\mathcal{N}_A(d)}{\text{ord}_d(2^k)}} = (NH(2^{k \cdot \text{ord}_d(2^k)}, 2^s))^{(1-\chi(d, 2^k))\frac{\mathcal{N}_A(d)}{\text{ord}_d(2^k)}}.$$

Case 3: $\chi(d, 2^k) = 1$. By [14, Lemma 4.5], every 2^k -cyclotomic class of A containing an element of order d is of type *III*. Then the number of cyclic codes D_l 's of length 2^s corresponding to d is

$$(N(2^{k \cdot \text{ord}_d(2^k)}, 2^s))^{\frac{\mathcal{N}_A(d)}{2\text{ord}_d(2^k)}} = (N(2^{k \cdot \text{ord}_d(2^k)}, 2^s))^{\chi(d, 2^k)\frac{\mathcal{N}_A(d)}{2\text{ord}_d(2^k)}}.$$

The desired result follows since d runs over all divisors of M . \square

This enumeration will be completed by counting the above numbers NE , NH , and N in Corollaries 4.5, 4.8, and 3.13, respectively.

2.2.2 Hermitian Case

We focus on the case where k is even. A 2^k -cyclotomic class $S_{2^k}(a)$ is said to be of *type I'* if $S_{2^k}(a) = S_{2^k}(-2^{\frac{k}{2}}a)$ or *type III'* if $S_{2^k}(a) \neq S_{2^k}(-2^{\frac{k}{2}}a)$. The primitive idempotent e induced by $S_{2^k}(a)$ is said to be of type $\lambda \in \{I', III'\}$ if $S_{2^k}(a)$ is a 2^k -cyclotomic class of type λ .

Without loss of generality, the representatives a_1, a_2, \dots, a_t of 2^k -cyclotomic classes can be chosen such that $\{a_i \mid i = 1, 2, \dots, r_{I'}\}$ and $\{a_{r_{I'}+j}, a_{r_{I'}+r_{III'}+j} = -2^{\frac{k}{2}}a_{r_{I'}+j} \mid j = 1, 2, \dots, r_{III'}\}$ are sets of representatives of 2^k -cyclotomic classes of types *I'* and *III'*, respectively, where $t = r_{I'} + 2r_{III'}$.

Rearranging the terms in the decomposition of \mathcal{R} in (2.1) based on the above 2 types primitive idempotents, we have

$$\mathbb{F}_{2^k}[A \times \mathbb{Z}_2] \cong \bigoplus_{i=1}^t (\mathcal{R}e_i)[\mathbb{Z}_2] \cong \left(\prod_{j=1}^{r_{I'}} \mathcal{S}_j \right) \times \left(\prod_{l=1}^{r_{III'}} (\mathcal{T}_l \times \mathcal{T}_l) \right), \quad (2.8)$$

where $\mathcal{S}_j := \mathbb{F}_{2^{k_j}} + u\mathbb{F}_{2^{k_j}}$ for all $j = 1, 2, \dots, r_{I'}$ and $\mathcal{T}_l := \mathbb{F}_{2^{k_{r_{I'}+l}}} + u\mathbb{F}_{2^{k_{r_{I'}+l}}}$ for all $l = 1, 2, \dots, r_{II'}$.

From (2.8), we have

$$\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}] \cong \left(\prod_{j=1}^{r_{I'}} \mathcal{S}_j[\mathbb{Z}_{2^s}] \right) \times \left(\prod_{l=1}^{r_{II'}} (\mathcal{T}_l[\mathbb{Z}_{2^s}] \times \mathcal{T}_l[\mathbb{Z}_{2^s}]) \right). \quad (2.9)$$

Hence, an abelian code C in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ can be viewed as

$$C \cong \left(\prod_{j=1}^{r_{I'}} C_j \right) \times \left(\prod_{l=1}^{r_{II'}} (D_l \times D'_l) \right), \quad (2.10)$$

where C_j , D_l and D'_l are cyclic codes in $\mathcal{S}_j[\mathbb{Z}_{2^s}]$, $\mathcal{T}_l[\mathbb{Z}_{2^s}]$ and $\mathcal{T}_l[\mathbb{Z}_{2^s}]$, respectively, for all $j = 1, 2, \dots, r_{I'}$ and $l = 1, 2, \dots, r_{II'}$.

Using the analysis similar to those in [16, Section II.D], the Hermitian dual of C in (2.10) is of the form

$$C^{\perp_H} \cong \left(\prod_{j=1}^{r_{I'}} C_j^{\perp_H} \right) \times \left(\prod_{l=1}^{r_{II'}} ((D'_l)^{\perp_E} \times D_l^{\perp_E}) \right).$$

Similar to [16, Corollary 2.8], necessary and sufficient conditions for an abelian code in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ to be Hermitian self-dual are now given using the notions of cyclic codes of length 2^s over \mathcal{S}_j and \mathcal{T}_l in the following corollary.

Corollary 2.3. *An abelian code C in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ is Hermitian self-dual if and only if in the decomposition (2.10),*

i) C_j is a Hermitian self-dual cyclic code of length 2^s over \mathcal{S}_j for all $j = 1, 2, \dots, r_{I'}$, and

ii) $D'_l = D_l^{\perp_E}$ is a cyclic code of length 2^s over \mathcal{T}_l for all $l = 1, 2, \dots, r_{II'}$.

Given a positive integer k and an odd positive integer j , the pair $(j, 2^k)$ is said to be *oddly good* if j divides $2^{kt} + 1$ for some odd integer $t \geq 1$, and *evenly good* if j divides $2^{kt} + 1$ for some even integer $t \geq 2$. These notions have been introduced in [16] for characterizing the Hermitian self-dual abelian codes in PIGAs.

Let λ be a function defined on the pair $(j, 2^k)$, where j is an odd positive integer, as

$$\lambda(j, 2^k) = \begin{cases} 0 & \text{if } (j, 2^k) \text{ is oddly good,} \\ 1 & \text{otherwise.} \end{cases} \quad (2.11)$$

The number of Hermitian self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ can be determined as follows.

Theorem 2.4. *Let k be an even positive integer and let s be a positive integer. Let A be a finite abelian group of odd order and exponent M . Then the number of Hermitian self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ is*

$$\prod_{d|M} \left(NH(2^{k \cdot \text{ord}_d(2^k)}, 2^s) \right)^{(1-\lambda(d, 2^{\frac{k}{2}})) \frac{\mathcal{N}_A(d)}{\text{ord}_d(2^k)}} \times \prod_{d|M} \left(N(2^{k \cdot \text{ord}_d(2^k)}, 2^s) \right)^{\lambda(d, 2^{\frac{k}{2}}) \frac{\mathcal{N}_A(d)}{2 \text{ord}_d(2^k)}},$$

where $\mathcal{N}_A(d)$ denotes the number of elements of order d in A determined in [1].

Proof. By Corollary 2.3 and (2.10), it is enough to determine the numbers cyclic codes C_j 's and D_l 's of length 2^s in (2.10) such that C_j is Hermitian self-dual.

The desired result can be obtained using arguments similar to those in the proof of Theorem 2.2, where [16, Lemma 3.5] is applied instead of [14, Lemma 4.5]. \square

This enumeration will be completed by counting the above numbers NH and N in Corollaries 4.8 and 3.13, respectively.

3 Cyclic Codes of Length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$

The enumeration of self-dual abelian codes in non-PIGAs in the previous section requires properties of cyclic codes of length 2^s over $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$. In this section, a more general situation is discussed. Precisely, properties cyclic of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are studied for all primes p . We note that algebraic structures of cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ was studied in [7] and [8]. Here, based on [20], we give an alternative characterization of such codes which is useful in studying self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$.

First, we note that there exists a one-to-one correspondence between the cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ and the ideals in the quotient ring $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$. Precisely, a cyclic code C of length p^s can be represented by the ideal

$$\left\{ \sum_{i=0}^{p^s-1} v_i x^i \mid (v_0, v_1, \dots, v_{p^s-1}) \in C \right\}$$

in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$.

Form now on, a cyclic code C will be referred to as the above polynomial presentation. Note that the map $\mu : (\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^n - 1 \rangle \rightarrow \mathbb{F}_{p^k}[x]/\langle x^n - 1 \rangle$ defined by

$$\mu(f(x)) = f(x) \pmod{u}$$

is a surjective ring homomorphism. For each cyclic code C in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ and $i \in \{0, 1\}$, let

$$\text{Tor}_i(C) = \{\mu(v(x)) \mid v(x) \in (\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^n - 1 \rangle \text{ and } u^i v(x) \in C\}.$$

For each $i \in \{0, 1\}$, $\text{Tor}_i(C)$ is called the i th *torsion code* of C . The codes $\text{Tor}_0(C) = \mu(C)$ and $\text{Tor}_1(C)$ are some time called the *residue* and *torsion codes* of C , respectively.

It is not difficult to see that for each $i \in \{0, 1\}$, $c(x) \in \text{Tor}_i(C)$ if and only if $u^i(c(x) + uz(x)) \in C$ for some $z(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$. Consequently, we have that $\text{Tor}_0(C) \subseteq \text{Tor}_1(C)$ are ideals in $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ (cyclic codes of length p^s over \mathbb{F}_{p^k}). We note that every ideal C in $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ is of the form $\langle (x - 1)^i \rangle$ for some $0 \leq i \leq p^s$ and the cardinality of C is p^{s-i} .

From the structures of cyclic codes of length p^s over \mathbb{F}_{p^k} discussed above and [8, Proposition 2.5], we have the following properties of the torsion and residue codes.

Proposition 3.1. *Let C be an ideal in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ and let $i \in \{0, 1\}$. Then the following statements hold.*

- (i) $\text{Tor}_i(C)$ is an ideal of $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ and $\text{Tor}_i(C) = \langle (x - 1)^{T_i} \rangle$ for some $0 \leq T_i \leq p^s$.
- (ii) If $\text{Tor}_i(C) = \langle (x - 1)^{T_i} \rangle$, then $|\text{Tor}_i(C)| = (p^k)^{p^s - T_i}$.
- (iii) $|C| = |\text{Tor}_0(C)| \cdot |\text{Tor}_1(C)| = (p^k)^{2p^s - (T_0 + T_1)}$.

With the notations given in Proposition 3.1, for each $i \in \{0, 1\}$, $T_i(C) := T_i$ is called the i th-*torsional degree* of C .

Remark 3.2. From Proposition 3.1 and the definition above, we have the following facts.

- (i) Since $\text{Tor}_0(C) \subseteq \text{Tor}_1(C)$, we have $0 \leq T_1(C) \leq T_0(C) \leq p^s$.
- (ii) If $u(x - 1)^t \in C$, then $t \geq T_1(C)$.

Next, we determine a generator set of an ideal in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$.

Theorem 3.3. *Let C be an ideal of $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$. Then*

$$C = \langle s_0(x), us_1(x) \rangle,$$

where, for each $i \in \{0, 1\}$,

- (i) either $s_j(x) = 0$ or $s_j(x) = (x - 1)^{t_j} + uz_j(x)$ for some $z_j(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ and $0 \leq t_j < p^s$,
- (ii) $s_j(x) \neq 0$ if and only if $\text{Tor}_j(C) \neq \{0\}$ and $\text{Tor}_0(C) \neq \text{Tor}_1(C)$, and
- (iii) if $s_j(x) \neq 0$, then $\text{Tor}_j(C) = \langle (x - 1)^{t_j} \rangle$.

Proof. The statement can be obtained using a slight modification of the proof of [11, Theorem 6.5]. For completeness, the details are given as follows.

For each ideal I in $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$, it can be represented as $I = \{0\}$ or $C = \langle (x-1)^{t_0} \rangle$ where $0 \leq t_0 < p^s$. If $I = \{0\}$, then we are done by choosing $s_0(x) = 0$. For $0 \leq t_0 < p^s$, let $s_0(x) = (x-1)^{t_0}$. By abuse of notation, $\text{Tor}_0(C) = C = \langle (x-1)^{t_0} \rangle$. Hence, $s_0(x)$ satisfies the conditions (i), (ii) and (iii).

Since C is an ideal of the ring $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ and μ is a surjective ring homomorphism, $\mu(C)$ is an ideal of $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ which implies that $\mu(C) = \langle s'_0(x) \rangle$ where $s'_0(x)$ satisfies the conditions (i), (ii) and (iii). If $s'_0(x) = 0$, then take $s_0(x) = 0$. Assume that $s'_0(x) \neq 0$, then $s'_0(x) = (x-1)^{t_0}$ where $0 \leq t_0 < p^s$. Then there exists $z_0(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ such that $s_0(x) = (x-1)^{t_0} + uz_0(x) \in C$ and $\mu(s_0(x)) = s'_0(x)$, i.e., $s_0(x) = s'_0(x) + uz_0(x)$. Since $\text{Tor}_1(C)$ is an ideal of $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$, it follows that $\text{Tor}_1(C) = \langle (x-1)^{t_1} \rangle$ for some $0 \leq t_1 \leq p^s$. Let $s_1(x) = (x-1)^{t_1}$. Claim that $C = \langle s_0(x), us_1(x) \rangle$. Since C is an ideal of $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$, we have $u(x-1)^{t_1} \in C$. Thus, $\langle s_0(x), us_1(x) \rangle \subseteq C$. To show that $C \subseteq \langle s_0(x), us_1(x) \rangle$, let $c(x) \in C$. Then $\mu(c(x)) = w(x)s'_0(x)$ for some $w(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$. Thus, $\mu(c(x)) = w(x)s'_0(x)$ which implies that $c(x) = w(x)s'_0(x) + uw'(x) = w(x)s_0(x) - uw(x)z_0(x) + uw'(x) = w(x)s_0(x) + u(-z_0(x)w(x) + w'(x))$ for some $w'(x) \in \text{Tor}_1(C)$. Since $w'(x) \in \text{Tor}_1(C)$, it follows that $c(x) \in \langle s_0(x), us_1(x) \rangle$. Therefore, $C = \langle s_0(x), us_1(x) \rangle$ as desired.

Note that $s_1(x) = (x-1)^{p^k} = 0$ implies $C = \{0\}$. Assume that $C \neq \{0\}$. Then $s_1(x) = (x-1)^{t_1}$ where $0 \leq t_1 < p^s$. If $s_0(x) = 0$, then we are done. Assume that $s_0(x) \neq 0$. Then $s_0(x) = (x-1)^t$ where $0 \leq t < p^s$. Hence, $\text{Tor}_0(C) = \langle (x-1)^t \rangle$. Since $\text{Tor}_0(C) \subseteq \text{Tor}_1(C)$, we have $t_1 \leq t$. If $t_1 < t$, then we are done. Assume that $t_1 = t$. Then $s_0(x) = (x-1)^t + uz_0(x)$ for some $z_0(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$. It follows that $us_1(x) = u(x-1)^{t_1} = u(x-1)^t = us_0(x) \in \langle s_0(x) \rangle$, a contradiction. Therefore, $C = \langle s_0(x) \rangle = \langle s_0(x), us_0(x) \rangle = \langle s_0(x), us_1(x) \rangle$. \square

However, the generator set given in Theorem 3.3 does not need to be unique. The unique presentation is given in the following theorem.

Theorem 3.4. *Let C be an ideal in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$, $T_0 := T_0(C)$ and $T_1 := T_1(C)$. Then*

$$C = \langle f_0(x), f_1(x) \rangle,$$

where

$$f_0(x) = \begin{cases} (x-1)^{T_0} + u(x-1)^{T_1}h(x) & \text{if } T_0 < p^s, \\ 0 & \text{if } T_0 = p^s, \end{cases}$$

and

$$f_1(x) = \begin{cases} u(x-1)^{T_1} & \text{if } T_1 < p^s, \\ 0 & \text{if } T_1 = p^s, \end{cases}$$

with $h(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ is either zero or a unit with $t + \deg(h(x)) < T_0$.

Moreover, $(f_0(x), f_1(x))$ is unique in the sense that if there exists a pair $(g_0(x), g_1(x))$ of polynomials satisfying the conditions in the theorem, then $f_0(x) = g_0(x)$ and $f_1(x) = g_1(x)$.

Proof. If $C = \{0\}$, then $\text{Tor}_1(C) = \{0\}$ and $\text{Tor}_0(C) = \{0\}$ which imply that $T_0 = p^s$ and $T_1 = p^s$. The polynomials $f_0(x) = 0$ and $f_1(x) = 0$ have the desired properties.

Next, assume that $C \neq \{0\}$. Then there exists the smallest nonnegative integer $r \in \{0, 1\}$ such that $T_r < p^s$. From Theorem 3.3, it can be concluded that

$$C = \langle s_0(x), us_1(x) \rangle,$$

where

$$s_0(x) = \begin{cases} (x-1)^{T_0} + ug(x) \text{ for some } g(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle & \text{if } r = 0, \\ 0 & \text{if } r = 1, \end{cases}$$

and

$$s_1(x) = (x-1)^{T_1}.$$

Case 1: $r = 0$. Then $s_0(x) = (x-1)^{T_0} + ug(x)$ for some $g(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ and $s_1(x) = (x-1)^{T_1}$. It follows that $C = \langle (x-1)^{T_0} + ug(x), (x-1)^{T_1} \rangle$, $T_1(C) = T_1$ and $T_0(C) = T_0$. Let $f_1(x) = us_1(x)$. Since $g(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$, we have

$$s_0(x) = (x-1)^{T_0} + u \sum_{j=0}^{p^s-1} a_j(x-1)^j,$$

where $a_j \in \mathbb{F}_{p^k}$ for all $j = 0, 1, \dots, p^k - 1$. Since $u(x-1)^{T_1} = us_1(x) \in C$, we have

$ua_j(x-1)^j \in C$ for all $j = T_1, T_1 + 1, \dots, p^s - 1$. It follows that $u \sum_{j=T_1}^{p^s-1} a_j(x-1)^j \in C$.

Let $f_0(x) = (x-1)^{T_0} + u \sum_{j=0}^{T_1-1} a_j(x-1)^j$. Then $f_0(x) = s_0(x) - u \sum_{j=T_1}^{p^s-1} a_j(x-1)^j \in C$.

We show that $C = \langle f_0(x), f_1(x) \rangle$. From the discussion above, we have $\langle f_0(x), f_1(x) \rangle \subseteq C$. Since $us_1(x) = u(x-1)^{T_1} = f_1(x)$, it follows that $ua_j(x-1)^j \in \langle f_0(x), f_1(x) \rangle$ for all $j = T_1, T_1 + 1, \dots, p^s - 1$. Hence, $u \sum_{j=T_1}^{p^s-1} a_j(x-1)^j \in \langle f_0(x), f_1(x) \rangle$ which implies that

$$s_0(x) = f_0(x) + u \sum_{j=T_1}^{p^s-1} a_j(x-1)^j \in \langle f_0(x), f_1(x) \rangle.$$

Therefore, $C = \langle s_0(x), us_1(x) \rangle \subseteq \langle f_0(x), f_1(x) \rangle$. As desired, $C = \langle f_0(x), f_1(x) \rangle$.

Case 2: $r = 1$. Then $s_0(x) = 0$ and $s_1(x) = u(x-1)^{T_1}$ which implies that $\text{Tor}_1(C) = \langle (x-1)^{T_1} \rangle$ and $\text{Tor}_0(C) = \{0\}$. By choosing $f_0(x) = 0$ and $f_1(x) = u(x-1)^{T_1}$, the result follows.

To prove the uniqueness, let $C = \langle g_0(x), g_1(x) \rangle$ be such that $g_0(x)$ and $g_1(x)$ satisfying the conditions in the theorem. Then $g_1(x) = u(x-1)^{T_1} = f_1(x)$.

Write $g_0(x) = (x-1)^{T_0} + u \sum_{j=0}^{T_1-1} c_j(x-1)^j$ where $c_j \in \mathbb{F}_{p^k}$. Then

$$f_0(x) - g_0(x) = u \sum_{j=0}^{T_1-1} (a_j - c_j)(x-1)^j.$$

It can be seen that $f_0(x) - g_0(x) = u(x-1)^l h(x)$, where $h(x) = 0$ or $h(x)$ is a unit with $l \leq T_1 - 1 < T_1$. If $h(x)$ is a unit, then $u(x-1)^l \in C$ which implies that $l \geq T_1$, a contradiction. Hence, $h(x) = 0$ which means that $f_0(x) = g_0(x)$ as desired. \square

Definition 3.5. For each ideal C in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$, denote by $C = \langle \langle f_0(x), f_1(x) \rangle \rangle$ the unique representation of the ideal C obtained in Theorem 3.4.

Illustrative examples of the representations in Theorem 3.3 and Theorem 3.4 are given as follows.

Example 3.6. Consider the ideal $C = \langle (x-1)^2 \rangle$ in $(\mathbb{F}_2 + u\mathbb{F}_2)[x]/\langle x^4 - 1 \rangle$. Using Theorem 3.4, we obtain that C has the unique representation $\langle \langle (x-1)^2, u(x-1)^2 \rangle \rangle$. Based on Theorem 3.3, C can be represented as $\langle (x-1)^2, 0 \rangle$, $\langle (x-1)^2 + u(x-1)^2, 0 \rangle$, and $\langle (x-1)^2 + u(x-1)^3, 0 \rangle$.

The annihilator of an ideal C in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ is key to determine properties C as well as the number of ideals in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$.

Definition 3.7. Let C be an ideal in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$. The *annihilator* of C , denoted by $\text{Ann}(C)$, is defined to be the set $\{f(x) \in (\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle \mid f(x)g(x) = 0 \text{ for all } g(x) \in C\}$.

The following properties of the annihilator can be obtained using arguments similar to those in the case of Galois rings in [20].

Theorem 3.8. Let C be an ideal of $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$. Then the following statements hold.

- (i) $\text{Ann}(C)$ is an ideal of $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$.
- (ii) If $|C| = (p^k)^d$, then $|\text{Ann}(C)| = (p^k)^{(2 \cdot p^s - d)}$.

(iii) $\text{Ann}(\text{Ann}(C)) = C$

Theorem 3.9. *Let \mathcal{I} denote the set of ideals of $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ and let $\mathcal{A} = \{C \in \mathcal{I} \mid T_0(C) + T_1(C) \leq p^s\}$ and $\mathcal{A}' = \{C \in \mathcal{I} \mid T_0(C) + T_1(C) \geq p^s\}$. Then the map $\phi : \mathcal{A} \rightarrow \mathcal{A}'$ defined by $C \mapsto \text{Ann}(C)$ is a bijection.*

The rest of this section is devoted to the determination of all ideals in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$. In view of Theorem 3.9, it suffices to focus on the ideals in \mathcal{A} .

For each $C = \langle \langle f_0(x), f_1(x) \rangle \rangle$ in \mathcal{A} , if $f_0(x) = 0$, then $T_0(C) = p^s$ and $T_1(C) = 0$. Hence, the only ideal in \mathcal{A} with $f_0(x) = 0$ is of the form $\langle \langle 0, u \rangle \rangle$. In the following two theorems, we assume that $f_0(x) \neq 0$.

Theorem 3.10. *Let $\langle \langle (x-1)^{i_0} + u(x-1)^t h(x), u(x-1)^{i_1} \rangle \rangle$ be the representation of an ideal in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$. Then it is a representation of an ideal in \mathcal{A} if and only if i_0, i_1, t are integers and $h(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ such that $0 \leq i_0 < p^s$, $0 \leq i_1 \leq \min\{i_0, p^s - i_0\}$, $t \geq 0$, $t + \deg(h(x)) < i_1$ and $h(x)$ is either zero or a unit in $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$.*

Proof. From Theorem 3.4, we have that i_0, i_1, t are integers and $h(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ such that $0 \leq i_0 < p^s$, $0 \leq i_1 \leq i_0$, $t \geq 0$, $t + \deg(h(x)) < i_1$ and $h(x)$ is either zero or a unit in $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$.

Assume that $\langle \langle (x-1)^{i_0} + u(x-1)^t h(x), u(x-1)^{i_1} \rangle \rangle$ is a representation of an ideal in \mathcal{A} . Then $i_0 + i_1 \leq p^s$ which implies that $i_1 \leq p^s - i_0$. Hence, we have $i_1 \leq \min\{i_0, p^s - i_0\}$.

Conversely, assume that $C = \langle \langle (x-1)^{i_0} + u(x-1)^t h(x), u(x-1)^{i_1} \rangle \rangle$, where i_0, i_1, t are integers and $h(x) \in \mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$ such that $0 \leq i_0 < p^s$, $0 \leq i_1 \leq \min\{i_0, p^s - i_0\}$, $t \geq 0$, $t + \deg(h(x)) < i_1$ and $h(x)$ is either zero or a unit in $\mathbb{F}_{p^k}[x]/\langle x^{p^s} - 1 \rangle$. Clearly, $i_0 + i_1 \leq p^s$. To show that $C = \langle \langle (x-1)^{i_0} + u(x-1)^t h(x), u(x-1)^{i_1} \rangle \rangle$ in \mathcal{A} and it remains to proof that $T_0(C) = i_0$ and $T_1(C) = i_1$.

Let $D = \langle (x-1)^{p^s-i_1} - u(x-1)^{p^s-i_0-i_1+t} h(x), u(x-1)^{p^s-i_0} \rangle$. It is not difficult to see that

$$D = \langle (x-1)^{p^s-i_1} + u(x-1)^{p^s-i_0} - u(x-1)^{p^s-i_0-i_1+t} h(x), u(x-1)^{p^s-i_0} \rangle.$$

Since

$$\begin{aligned} ((x-1)^{i_0} + u(x-1)^{i_1} h(x)) ((x-1)^{p^s-i_1} + u(x-1)^{p^s-i_0} - u(x-1)^{p^s-i_0-i_1+t} h(x)) &= 0, \\ ((x-1)^{i_0} + u(x-1)^{i_1} h(x)) (u(x-1)^{p^s-i_0}) &= 0, \end{aligned}$$

and

$$(u(x-1)^{i_1}) ((x-1)^{p^s-i_1} + u(x-1)^{p^s-i_0} - u(x-1)^{p^s-i_0-i_1+t} h(x)) = 0,$$

we have $D \subseteq \text{Ann}(C)$. By Proposition 3.1, we obtain that $T_j(C) \leq i_j$ and $T_j(D) \leq p^{s-i_1-j}$ for all $j \in \{0, 1\}$. Hence, $|C| \geq (p^k)^{2 \cdot p^s - i_0 - i_1}$ and $|\text{Ann}(C)| \geq |D| \geq (p^k)^{i_0 + i_1}$. Since $|C| \cdot |\text{Ann}(C)| = (p^k)^{2 \cdot p^s}$, we have $\text{Ann}(C) = D$. Therefore, $T_0(C) = i_0$, $T_1(C) = i_1$ as desired. \square

Since every polynomial $\sum_{i=0}^m a_i(x-1)^i$ in $\mathbb{F}_{p^k}[x]$ is either 0 or $(x-1)^t h(x)$, where $h(x)$ is a unit in $\mathbb{F}_{p^k}[x]$ and $0 \leq t \leq m - \deg(h(x))$, Theorem 3.10 is rewritten as follows:

Theorem 3.11. *The expression $\langle \langle (x-1)^{i_0} + u \sum_{j=0}^{i_1-1} h_j(x-1)^j, u(x-1)^{i_1} \rangle \rangle$ represents an ideal in \mathcal{A} if and only if i_0 and i_1 are integers such that $0 \leq i_0 < p^s$, $0 \leq i_1 \leq \min\{i_0, p^s - i_0\}$, $i_0 + i_1 \leq p^s$, and $h_j \in \mathbb{F}_{p^k}$ for all $0 \leq j < i_1$.*

The number of distinct ideals of $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ of a fixed $d = T_0 + T_1$ is given in the following proposition.

Proposition 3.12. *Let $0 \leq d \leq p^s$. Then the number of distinct ideals in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ with $T_0 + T_1 = d$ is*

$$\frac{p^{k(K+1)} - 1}{p^k - 1},$$

where $K = \min\{\lfloor \frac{d}{2} \rfloor, p^s - \lfloor \frac{d}{2} \rfloor\}$.

Proof. Let $T_1 = i_1$ and $i_0 := T_0 = d - T_1$ be fixed.

Case 1 : $d < p^s$. Then $i_0 \leq i_0 + i_1 = T_0 + T_1 = d < p^s$. By Theorem 3.11, it follows that $C = \langle \langle (x-1)^{i_0} + u \sum_{j=0}^{i_1-1} h_j(x-1)^j, u(x-1)^{i_1} \rangle \rangle$. Then the choices for $\sum_{j=0}^{i_1-1} h_j(x-1)^j$ is $(p^k)^{i_1}$. By Theorem 3.11 again, we also have $T_1 \leq \min\{T_0, p^s - T_0\}$. Since $T_0 + T_1 = d$, we obtain that $T_1 \leq \lfloor \frac{d}{2} \rfloor \leq T_0$, and hence, $T_1 \leq \min\{\lfloor \frac{d}{2} \rfloor, p^s - T_0\} \leq \min\{\lfloor \frac{d}{2} \rfloor, p^s - \lfloor \frac{d}{2} \rfloor\}$. Now, vary T_1 from 0 to K , we obtain that there are $1 + p^k + \dots + (p^k)^K = \frac{p^{k(K+1)} - 1}{p^k - 1}$ ideals with $T_0 + T_1 = d$.

Case 2 : $d = p^s$. If $i_0 = p^s$, then the only ideal with $T_0 + T_1 = p^s$ is the ideal represented by $\langle \langle 0, u \rangle \rangle$. If $i_0 < p^s$, then we have $p^k + (p^k)^2 \dots + (p^k)^K$ ideals by arguments similar to those in Case 1. \square

For a cyclic code C in \mathcal{A} , we have $C \neq \text{Ann}(C)$ whenever $T_0(C) + T_1(C) < p^s$. In the case where $T_0(C) + T_1(C) = p^s$, by the proof of Theorem 3.10, the annihilator of the cyclic code $C = \langle \langle (x-1)^{i_0} + u(x-1)^t h(x), u(x-1)^{i_1} \rangle \rangle$ is of the form $\text{Ann}(C) = \langle \langle (x-1)^{i_0} - u(x-1)^t h(x), u(x-1)^{i_1} \rangle \rangle$. If p is odd, then $C = \text{Ann}(C)$ occurs only the case $h(x) = 0$. In the case where $p = 2$, $C = \text{Ann}(C)$ is always true. By Proposition 3.12 and the bijection given in Theorem 3.9, the number of cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ can be summarized as follows.

Corollary 3.13. *The number of cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ is*

$$N(p^k, p^s) = \begin{cases} 2 \left(\sum_{d=0}^{p^s} \frac{p^{k(\min\{\lfloor \frac{d}{2} \rfloor, p^s - \lfloor \frac{d}{2} \rfloor + 1\}) - 1}}{p^k - 1} \right) - \frac{p^{k(p^s - 1 + 1)} - 1}{p^k - 1} & \text{if } p = 2, \\ 2 \left(\sum_{d=0}^{p^s} \frac{p^{k(\min\{\lfloor \frac{d}{2} \rfloor, p^s - \lfloor \frac{d}{2} \rfloor + 1\}) - 1}}{p^k - 1} \right) - 1 & \text{if } p \text{ is odd.} \end{cases}$$

Proof. From Theorem 3.9, the number of cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ is $|\mathcal{A} \cup \mathcal{A}'| = |\mathcal{A}| + |\mathcal{A}'| - |\mathcal{A} \cap \mathcal{A}'|$. The desired results follow immediately from the discussion above. \square

4 Self-Dual Cyclic Codes of Length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$

In this section, characterization and enumeration self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are given under the Euclidean and Hermitian inner products.

4.1 Euclidean Self-Dual Cyclic Codes of Length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$

Characterization and enumeration of Euclidean self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are given in this subsection.

For each subset A of $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$, denote by \overline{A} the set of polynomials $\overline{f(x)}$ for all $f(x)$ in A , where $\overline{}$ is viewed as the conjugation on the group ring $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[\mathbb{Z}_{p^s}]$ defined in Section 2. From the definition of the annihilator, the next theorem can be derived similar to [9, Proposition 2.12].

Theorem 4.1. *Let C be an ideal in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$. Then $C^{\perp_E} = \overline{\text{Ann}(C)}$.*

Using the unique generators of an ideal C in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ determined in Theorem 3.11, the Euclidean dual of C can be given in the following theorem.

Theorem 4.2. *Let $C = \left\langle \left\langle (x-1)^{i_0} + u \sum_{j=0}^{i_1-1} h_j(x-1)^j, u(x-1)^{i_1} \right\rangle \right\rangle$ be an ideal in \mathcal{A} , where $h_j \in \mathbb{F}_{p^k}$ for all $0 \leq j \leq i_1 - 1$. Then C^{\perp_E} is in the form of*

$$C^{\perp_E} = \left\langle \left\langle (x-1)^{p^s-i_1} - u(x-1)^{p^s-i_0-i_1} \left(\sum_{r=0}^{i_1-1} \sum_{j=0}^r (-1)^{i_0+j} \binom{i_0-j}{r-j} h_j(x-1)^r \right), \right. \right. \\ \left. \left. u(x-1)^{p^s-i_0} \right\rangle \right\rangle. \quad (4.1)$$

Proof. From the proof of Theorem 3.10, we have

$$\text{Ann}(C) = \left\langle \left\langle (x-1)^{p^s-i_1} - u(x-1)^{p^s-i_0-i_1} \sum_{j=0}^{i_1-1} h_j(x-1)^j, u(x-1)^{p^s-i_0} \right\rangle \right\rangle.$$

By Theorem 4.1, it follows that $C^{\perp_E} = \overline{\text{Ann}(C)}$. Hence, C^{\perp_E} contains the elements $u(x-1)^{p^s-i_0}$ and

$$(x-1)^{p^s-i_1} - u(x-1)^{p^s-i_0-i_1} \sum_{j=0}^{i_1-1} (-1)^{i_0+j} h_j(x-1)^j x^{i_0-j}.$$

By writing $x = (x-1) + 1$ and using the Binomial Theorem, it follows that C^{\perp_E} contains the element

$$(x-1)^{p^s-i_1} - u(x-1)^{p^s-i_0-i_1} \left(\sum_{j=0}^{i_1-1} \left(\sum_{l=0}^{i_0-j} (-1)^{i_0+j} \binom{i_0-j}{l} h_j(x-1)^{i_0-l} \right) \right).$$

Hence,

$$\left\langle \left\langle (x-1)^{p^s-i_1} - u(x-1)^{p^s-i_0-i_1} \left(\sum_{j=0}^{i_1-1} \left(\sum_{l=0}^{i_0-j} (-1)^{i_0+j} \binom{i_0-j}{l} h_j(x-1)^{i_0-l} \right) \right) \right\rangle \right\rangle, \\ (x-1)^{p^s-i_1} \rangle \subseteq C^{\perp_E}.$$

By counting the number of elements, the two sets are equal as desired. Updating the indices, it can be concluded that

$$C^{\perp_E} = \left\langle \left\langle (x-1)^{p^s-i_1} - u(x-1)^{p^s-i_0-i_1} \left(\sum_{r=0}^{i_1-1} \left(\sum_{j=0}^r (-1)^{i_0+j} \binom{i_0-j}{r-j} h_j(x-1)^r \right) \right) \right\rangle \right\rangle, \\ (x-1)^{p^s-i_1} \rangle.$$

□

Assume that an ideal $C = \left\langle \left\langle (x-1)^{i_0} + u \sum_{j=0}^{i_1-1} h_j(x-1)^j, u(x-1)^{i_1} \right\rangle \right\rangle$ in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ is Euclidean self-dual, i.e., $C = C^{\perp_E}$. By Theorem 4.1, it is equivalent to $p^s = i_0 + i_1$ and

$$-h_t = \sum_{j=0}^t (-1)^{i_0+l} \binom{i_0-j}{r-j} h_j \quad (4.2)$$

in \mathbb{F}_{p^k} for all $0 \leq t \leq i_i - 1$.

We note that, if $i_1 = 0$, then it is not difficult to see that only the ideal generated by u is Euclidean self-dual.

For the case $i_1 \geq 1$, the situation is more complicated. First, we recall an $i_1 \times i_1$ matrix $M(p^s, i_1)$ over \mathbb{F}_{p^k} defined in [19] as

$$M(p^s, i_1) = \begin{bmatrix} (-1)^{i_0} + 1 & 0 & 0 & \dots & 0 \\ (-1)^{i_0} \binom{i_0}{1} & (-1)^{i_0+1} + 1 & 0 & \dots & 0 \\ (-1)^{i_0} \binom{i_0}{2} & (-1)^{i_0+1} \binom{i_0-1}{1} & (-1)^{i_0+2} + 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^{i_0} \binom{i_0}{i_1-1} & (-1)^{i_0+1} \binom{i_0-1}{i_1-2} & (-1)^{i_0+2} \binom{i_0-2}{i_1-3} & \dots & (-1)^{i_0+i_1-1} + 1 \end{bmatrix}. \quad (4.3)$$

It is not difficult to see that i_1 equations from (4.2) are equivalent to the matrix equation

$$M(p^s, i_1) \mathbf{h} = \mathbf{0} \quad (4.4)$$

where $\mathbf{h} = (h_0, h_1, \dots, h_{i_1-1})^T$ and $\mathbf{0} = (0, 0, \dots, 0)^T$.

Moreover, it can be concluded that the ideal C is Euclidean self-dual if and only if $p^s = i_0 + i_1$ and $h_0, h_1, \dots, h_{i_1-1}$ satisfy (4.4). Since $h_0 = h_1 = \dots = h_{i_1-1} = 0$ is a solution of 4.4, the corresponding idea $\langle \langle (x-1)^{p^s-i_1}, u(x-1)^{i_1} \rangle \rangle$ is Euclidean self-dual in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$. Hence, for a fixed first torsion degree $1 \leq i_1 \leq p^s$, a Euclidean self-dual ideal in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ always exists. By solving (4.4), all Euclidean self-dual ideals in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ can be constructed. Therefore, for a fixed first torsion degree $1 \leq i_1 \leq p^s$, the number of Euclidean self-dual ideals in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ equals the number of solutions of (4.4) which is $p^{k\kappa}$, where κ is the nullity of $M(p^s, i_1)$ determined in [19].

Proposition 4.3 ([19, Proposition 3.3]). *Let κ be the nullity of $M(p^s, i_1)$. Then*

$$\kappa = \begin{cases} \lfloor \frac{i_1}{2} \rfloor & \text{if } p \text{ is odd;} \\ \lceil \frac{i_1+1}{2} \rceil & \text{if } p = 2. \end{cases}$$

The number of Euclidean self-dual cyclic codes in $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$ with first torsional degree i_1 is given in terms of the nullity of $M(p^s, i_1)$ as follows.

Proposition 4.4. *Let $i_1 > 0$ and let κ be the nullity of $M(p^s, i_1)$ over \mathbb{F}_{p^k} . Then the number of Euclidean self-dual cyclic codes of length p^s over \mathbb{F}_{p^k} with first torsional degree i_1 is*

$$(p^k)^\kappa.$$

From Theorem 3.10, we have $0 \leq i_1 \leq \lfloor \frac{p^s}{2} \rfloor$ since $i_0 + i_1 = p^s$. Hence, the number of Euclidean self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ is given by the following corollary.

Corollary 4.5. *Let p be a prime and let s and k be positive integers. Then the following statements hold.*

(i) *If p is odd, then the number of Euclidean self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ is*

$$NE(p^k, p^s) = \begin{cases} 2 \left(\frac{(p^k)^{\frac{p^s+1}{4}} - 1}{p^k - 1} \right) & \text{if } p^s \equiv 3 \pmod{4}, \\ 2 \left(\frac{(p^k)^{\frac{p^s-1}{4}} - 1}{p^k - 1} \right) + (p^k)^{\frac{p^s-1}{4}} & \text{if } p^s \equiv 1 \pmod{4}. \end{cases}$$

(ii) *If $p = 2$, then the number of Euclidean self-dual cyclic codes of length 2^s over $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$ is*

$$NE(p^k, p^s) = \begin{cases} 1 + 2^k & \text{if } s = 1, \\ 1 + 2^k + (2^k)^2 & \text{if } s = 2, \\ 1 + 2^k + 2(2^k)^2 \left(\frac{(2^k)^{(2^s-2)-1} - 1}{2^k - 1} \right) & \text{if } s \geq 3. \end{cases}$$

Proof. From Propositions 4.3 and 4.4, the number of Euclidean self-dual cyclic codes of length 2^s over $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$ is $\sum_{i=1=0}^{\lfloor \frac{p^s}{2} \rfloor} (p^k)^{\lfloor \frac{i+1}{2} \rfloor}$. Apply a suitable geometric sum, the results follow. \square

4.2 Hermitian Self-Dual Cyclic Codes of Length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$

Under the assumption that k is even, characterization and enumeration Hermitian self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are given in this section.

For a subset A of $(\mathbb{F}_{p^k} + u\mathbb{F}_{p^k})[x]/\langle x^{p^s} - 1 \rangle$, let

$$\rho(A) := \left\{ \sum_{i=0}^{p^s-1} \rho(a_i) x^i \mid \sum_{i=0}^{p^s-1} a_i x^i \in A \right\},$$

where $\rho(a + ub) = a^{p^{\frac{k}{2}}} + ub^{p^{\frac{k}{2}}}$.

Based on the structural characterization of C given in Theorem 3.11, the Hermitian dual of C is determined as follows.

Theorem 4.6. *Let C be an ideal in \mathcal{A} and*

$$C = \left\langle \left\langle (x-1)^{i_0} + u \sum_{j=0}^{i_1-1} h_j (x-1)^j, u(x-1)^{i_1} \right\rangle \right\rangle,$$

where $h_j \in \mathbb{F}_{p^k}$. Then C^{\perp_H} has the representation

$$C^{\perp_H} = \left\langle \left\langle (x-1)^{p^s-i_1} - u(x-1)^{p^s-i_0-i_1} \left(\sum_{r=0}^{i_1-1} \sum_{j=0}^r (-1)^{i_0+j} \binom{i_0-j}{r-j} h_j^{p^{k/2}} (x-1)^r \right), \right. \right. \\ \left. \left. u(x-1)^{p^s-i_0} \right\rangle \right\rangle.$$

Proof. From Theorem 4.2 and the fact that $C^{\perp_H} = \rho(C^{\perp_E})$, the result follows. \square

Assume that C is Hermitian self-dual. Then $C = C^{\perp_H}$ which implies that $|C| = (p^k)^{p^s}$ and $i_0 + i_1 = p^s$.

If $i_1 = 0$, then it is not difficult to see that the ideal generated by u is only Hermitian self-dual cyclic code of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$.

Assume that $i_1 \geq 1$. Then

$$-uh_t^{p^{k/2}} = u \sum_{j=0}^t (-1)^{i_0+j} \binom{i_0-j}{t-j} h_j$$

for all $0 \leq t \leq i_1 - 1$.

From i_1 equations above and the definition of $M(p^s, i_1)$, we have

$$M(p^s, i_1)\mathbf{h} + (\mathbf{h}^{p^{k/2}} - \mathbf{h}) = \mathbf{0} \quad (4.5)$$

where $\mathbf{h} = (h_1, h_2, \dots, h_{i_1})$, $\mathbf{h}^{p^{k/2}} = (h_1^{p^{k/2}}, h_2^{p^{k/2}}, \dots, h_{i_1}^{p^{k/2}})$ and $\mathbf{0} = (0, 0, \dots, 0)$.

From Theorem 3.10, we have $0 \leq i_1 \leq \lfloor \frac{p^s}{2} \rfloor$ since $i_0 + i_1 = p^s$.

Proposition 4.7. *Let k be an even positive integer and let i_1 be a positive integer such that $i_1 \leq \lfloor \frac{p^k}{2} \rfloor$. Then the number of solution of (4.5) in $\mathbb{F}_{p^k}^{i_1}$ is*

$$p^{ki_1/2}.$$

Proof. Let $\Psi : \mathbb{F}_{p^k} \rightarrow \mathbb{F}_{p^k}$ defined by $\alpha \mapsto \alpha^{p^{k/2}} - \alpha$ for all $\alpha \in \mathbb{F}_{p^k}$. Using the fact that $\Psi(1) = 0 = \Psi(0)$ and arguments similar to those in [15, Proposition 3.3], the result follows. \square

For a prime number p , a positive integer s and an even positive integer k , the number of Hermitian self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ can be determined in the following corollary.

Corollary 4.8. *Let p be a prime and let s and k be positive integers such that k is even. Then the number of Hermitian self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ is*

$$NH(p^k, p^s) = \sum_{i_1=0}^{\lfloor \frac{p^s}{2} \rfloor} p^{ki_1/2} = \frac{(p^{k/2})^{\lfloor \frac{p^s}{2} \rfloor + 1} - 1}{p^{k/2} - 1}.$$

5 Conclusions and Remarks

Euclidean and Hermitian self-dual abelian codes in non-PIGAs $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ are studied. The complete characterization and enumeration of such abelian codes are given and summarized as follows.

In Corollaries 2.1 and 2.3, self-dual abelian code in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ are shown to be a suitable Cartesian product of cyclic codes, Euclidean self-dual cyclic codes, and Hermitian self-dual cyclic codes of length 2^s over some Galois extension of the ring $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$. Subsequently, the characterizations and enumerations of cyclic and self-dual cyclic codes of length p^s over $\mathbb{F}_{p^k} + u\mathbb{F}_{p^k}$ are studied for all primes p . Combining these results, the following enumerations of Euclidean and Hermitian self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ are rewarded.

For each abelian group A of odd order and positive integers s and k , the number of Euclidean self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ is given in Theorem 2.2 in terms of the numbers N , NE , and NH of cyclic codes, Euclidean self-dual cyclic codes, and Hermitian self-dual cyclic codes of length 2^s over a Galois extension of $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$, respectively.

In addition, if k is even, the number of Hermitian self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ is given in Theorem 2.4 in terms of the numbers N and NH of cyclic codes and Hermitian self-dual cyclic codes of length 2^s over a Galois extension of $\mathbb{F}_{2^k} + u\mathbb{F}_{2^k}$, respectively.

We note that all numbers N , NE , and NH are determined in Corollaries 3.13, 4.5, and 4.8, respectively. Therefore, the complete enumerations of Euclidean and Hermitian self-dual abelian codes in $\mathbb{F}_{2^k}[A \times \mathbb{Z}_2 \times \mathbb{Z}_{2^s}]$ are established.

One of the interesting problems concerning the enumeration of self-dual abelian codes in $\mathbb{F}_2^k[A \times B]$, where A is an abelian group of odd order, is the case where B is a 2-group of other types.

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